CHEBYSHEV POLYNOMIAL APPROXIMATION OF THE ROOTS OF THE TRANSCENDENTAL EQUATION $\mu \tan \mu = b$

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A polynomial approximation of the roots of the equation $\mu \tan \mu = b$ are tabulated, valid uniformly on the interval $b \in [0, \infty]$.

A solution of series problems of mathematical physics is obtained in the form of an expansion in terms of the eigenfunctions which correspond to eigenvalues appearing as roots of the characteristic equation:

$$\mu \operatorname{tg} \mu = b, \quad 0 \leqslant b < \infty. \tag{1}$$

In particular, equation (1) occurs in connection with the solution of various problems of nonstationary thermal conductivity of a plane wall [1-4] (the constant b in this case represents the Biot number). Also, equation (1) is associated with the solution of the problem of elastic vibrations of a rod with a weight on its end [5], and with a series of other problems of the theory of vibrations and the theory of elasticity [6, 7].

Equation (1), being transcendental, has no analytic solution. The first six roots of equation (1) were tabulated in [1, 4, 6]; however, these tables are rarely applied [8] in calculations on an electronic digital computer at present, because the tables must be loaded into the computer memory and much computer time expended on scanning the tables and interpolating. It is significantly more economical to represent the functions by simple approximating expressions, which are easily programmed and rapidly calculated.

TABLE 1. Values of the Coefficients

k	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$
1	+1,731190	+0,9995033	+0,9960347	+0,9949041	+ 0,9943011
2	+0,01803645	+0,6869336	+0,9387871	+1,027060	+ 1,073294
3	−0,1228113	+0,2366123	−0,4041464	−0,6786636	− 0,8406609
4	+0,3632877	−0,2655127	+2,869571	+4,458086	+ 5,420388
5	−0,6424595	+0,5164442	−6,014370	−9,516726	−11,65410
6	+0,1557248	−1,120717	+4,106981	+7,093294	+ 8,944093
7	+0,06784049	+0,5175449	−0,9220110	−1,807099	− 2,366449
max. abs error	$2 \cdot 10^{-5}$	$2 \cdot 10^{-5}$	$6 \cdot 10^{-5}$	$7 \cdot 10^{-5}$	$7 \cdot 10^{-5}$

k	$i=6$	$i=7$	$i=8$	$i=9$	$i=10$
1	+ 0,9938999	+ 0,9936069	+ 0,9933831	+ 0,9932056	+ 0,9930676
2	+ 1,102344	+ 1,122498	+ 1,137306	+ 1,148728	+ 1,157671
3	− 0,9512002	− 1,032451	− 1,094453	− 1,143909	− 1,183316
4	+ 6,073552	+ 6,548281	+ 6,907961	+ 7,191839	+ 7,418709
5	−13,10170	−14,14980	−14,94200	−15,56496	−16,06331
6	+10,20501	+11,12051	+11,81383	+12,35951	+12,79686
7	− 2,751041	− 3,031781	− 3,245163	− 3,413537	− 3,548804
max. abs. error	$8 \cdot 10^{-5}$	$8 \cdot 10^{-5}$	$8 \cdot 10^{-5}$	$8 \cdot 10^{-5}$	$8 \cdot 10^{-5}$

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An optimal choice for this purpose is the approximation by uniform Chebyshev polynomials, for which a good generating algorithm has been developed [9].

For applications to a electronic digital computer, a Chebyshev approximation of the i-th root of equation (1) is proposed, in the form of a polynomial:

$$\mu_i = \pi(i-1) + \sum_{k=1}^{7} \alpha_{i,k} z^k, \tag{2}$$

where

$$z = \begin{cases} \sqrt{\dfrac{b}{3+b}} & \text{for} \quad i = 1, \\ \dfrac{b}{\pi(i-1)+b} & \text{for} \quad i \geqslant 2. \end{cases}$$

The coefficients $\alpha_{i,k}$ are determined on the basis of the condition of minimization of the maximum deviation between the exact value of the root and the approximation (2), for $z \in [0, 1]$. The solution of the minimization problem has been carried out with the aid of E. Y. Remeza's second algorithm [9].

The above table contains the coefficients of the polynomial (2) and the maximum value of the absolute error of the approximation for the first ten roots of equation (1). Maximum values of the approximating error were determined in the process of verification, carried out for z = 0 (0.005) 1; control values of the roots of equation (1) were evaluated with an accuracy no worse than 10^{-6}.

In conclusion, we observe that the results given by the polynomial (2) are in complete agreement with the values of the roots tabulated in [1].

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